Part – 6: Machine Design

Part 6.1: Theory of Failures

6.1.1 Theories of Failure under Static Load

The strength of machine members is based upon the mechanical properties of the materials used. Since these properties are usually determined from simple tension or compression tests, predicting failure in members subjected to uniaxial stress is both simple and straight-forward.

But the problem of predicting the failure stresses for members subjected to bi-axial or tri-axial stresses is much more complicated, that a large number of different theories have been formulated. The principle theories of failure for a member subjected to tri-axial stress are as follows:

1. Maximum principle (or normal) stress theory (also known as Rankine's theory).
2. Maximum shear stress theory (also known as Guest's or Tresca's theory).
3. Maximum principle (or normal) strain theory (also known as Saint Venant theory).
4. Maximum strain energy theory (also known as Haigh's theory).
5. Maximum distortion energy theory (also known as Hencky and Von Mises theory).
6. Octahedral Shearing Stress theory.

Ductile materials have identifiable yield strength that is often same in compression as in tension ($Syt = Syc = Sy$).

Brittle materials, do not exhibit identifiable yield strength, and are typically classified by ultimate tensile and compressive strengths, Sut and Suc, respectively (where Suc is given as a positive quantity)

Maximum principle or Normal Stress Theory (Rankine's Theory) for Brittle materials

The elastic failure or yielding occurs at a point in a member when the maximum principle or normal stress reaches the limiting strength of the material in a simple tension test irrespective of the value of other two principle stresses, i.e., when $\sigma_1 = \sigma_e$

Since the limiting strength for ductile materials is yield point stress and for brittle materials is ultimate stress, the maximum principle or normal stress (σ_1) is given by

$$\sigma_1 = \frac{\sigma_{yt}}{FOS} \quad For\ Ductile\ materials$$

$$= \frac{\sigma_u}{FOS} \quad For\ Brittle\ materials$$

Where, σ_{yt} = Yield stress in tension as determined from simple tension test

σ_u = Ultimate stress

FOS = Factor of Safety

Since this theory ignores the possibility of failure due to shearing stress, it is not used for ductile materials.

However, for brittle materials which are relatively strong in shear but weak in tension or compression, this theory is generally used.

Maximum Shear Stress Theory (Guest's or Tresca's Theory) for ductile materials.

The elastic failure occurs when the greatest shear stress reaches a value equal to the shear stress at elastic limit in a simple tension test.

$\frac{1}{2}(\sigma_1 - \sigma_3) = \frac{1}{2}\sigma_e$ or

$(\sigma_1 - \sigma_3) = \sigma_e$

Maximum Principle Strain Theory (Saint Venant's Theory)

The elastic failure occurs when the greatest principle (or normal) strain reaches the elastic limit point (*i.e.* strain at yield point) as determined from a simple tensile test.

According to the above theory, the elastic failure will occur, when

$$\frac{1}{E}[\sigma_1 - \nu(\sigma_2 + \sigma_3)] = \frac{\sigma_e}{E}$$

This theory over-estimates the elastic strength of ductile materials.

Maximum Strain Energy Theory (Beltrami's or Haigh's Theory) for ductile materials

The failure or yielding occurs when the strain energy per unit volume in a strained material reaches the limiting strain energy (*i.e.* strain energy at the yield point) per unit volume as determined from simple tension test.

According to this theory, the maximum energy which a body can store without deforming plastically is constant for that material irrespective of the manner of loading.

$$\frac{1}{2E}[\sigma_1{}^2 + \sigma_2{}^2 + \sigma_3{}^2 - 2\nu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)] = \frac{\sigma_e{}^2}{2E}$$

$$[\sigma_1{}^2 + \sigma_2{}^2 + \sigma_3{}^2 - 2\nu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)] = \sigma_e{}^2$$

This theory breaks down for a case when,

$\sigma_1 = \sigma_2 = \sigma_3 = -\sigma$

And in that case failure is predicted when

$$\sigma = \frac{\sigma_e}{\sqrt{3(1-2\nu)}}$$

But in fact with this type of loading (i.e.,) when there is uniform pressure all round (hydrostatic pressure), no failure occurs.

This theory may be used for ductile materials.

Shear Strain energy or Maximum Distortion Energy Theory (Hencky and Von Mises Theory)

The failure or yielding occurs at a point in a member when the distortion strain energy (also called shear strain energy) per unit volume in the stressed material reaches the limiting distortion energy (*i.e.* distortion energy at yield point) per unit volume as determined from a simple tension test. Mathematically, the maximum distortion energy theory for yielding is expressed as

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 2{\sigma_e}^2$$

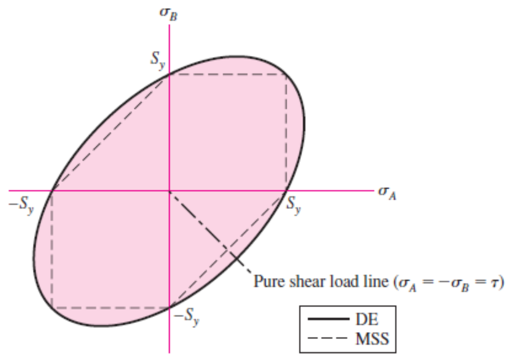

Fig. 6.1.1 The distortion-energy (DE) theory for plane stress states

This theory is mostly used for ductile materials in place of maximum strain energy theory.

Note: The maximum distortion energy is the difference between the total strain energy and the strain energy due to uniform stress.

Octahedral Shearing Stress Theory

According to this theory, the critical quantity is the shearing stress on the octahedral plane. The plane which is equally inclined to all the three principle axes is called the octahedral plane.

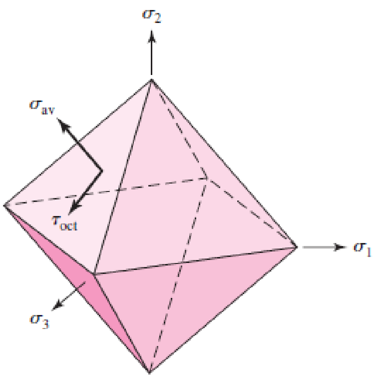

Fig. 6.1.2 Octahedral surfaces

$$\tau_{oct} = \frac{1}{3}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]^{\frac{1}{2}}$$

Where,

τ_{oct} = Octahedral shearing stress

Failure is said to occur when $\tau_{oct} = \frac{\sqrt{2}}{3}\sigma_e$

This theory is supported quite well by experimental evidences and is identical to Von Mises theory.

6.1.2 Theories of failure for two dimensional stresses:

Taking σ_3 as zero, the above equations reduce to

1. Maximum principle stress theory
   $\sigma_1 = \sigma_e$

2. Maximum principle strain theory
   $(\sigma_1 - \nu\sigma_2) = \sigma_e$

3. Maximum shear stress theory
   (a) For like tensile stresses
   $\sigma_1 > \sigma_2 > 0$
   $\sigma_1 - 0 = \sigma_e$
   $\sigma_1 = \sigma_e$

   (b) For unlike stresses $\sigma_1 \geq 0 > \sigma_2 (\sigma_1 \textit{tensile and } \sigma_2 \textit{ compressive})$
   $\sigma_1 - \sigma_2 = \sigma_e$
   $\textit{when } \sigma_2 \textit{ is tensile and } \sigma_1 = \textit{ compressive}$
   $\sigma_2 > 0 > \sigma_1$
   $\sigma_2 - \sigma_1 = \sigma_e$

4. Maximum strain energy theory
   ${\sigma_1}^2 + {\sigma_2}^2 - 2\nu\sigma_1\sigma_2 = {\sigma_e}^2$

5. Maximum distortion energy theory
   ${\sigma_1}^2 + {\sigma_2}^2 - \sigma_1\sigma_2 = {\sigma_e}^2$

6.1.3 Significance of theories of failure

Mode of failure of a ductile material differs from that of brittle material. It depends on a large number of factors like

- Nature and Properties of the material
- Type of loading
- Shape of member
- Temperature of member, etc.

If the loading conditions are suitably altered, a brittle material may be made to yield before failure. Hence, design of a member requires the determination of the mode of failure (yielding or fracture), and the factor (such as stress, strain and energy) associated with it. Full scale tests simulating all conditions would be ideal but not practicable.

In practice, in complex loading conditions, the factor associated with failure has to be identified and precautions taken to ensure that this factor does not exceed maximum allowable value determined on the basis of suitable tests (uniform tension or torsion) on the material in the laboratory.

Results of many laboratory tests on ductile material shows shear stress from torsion tests varies between 0.55 and 0.6 of the yield strength determined from tension tests. This result agrees with shear strain energy theory and octahedral shear stress theory. The maximum shear stress theory predicts that the shear yield value is 0.5 times the tensile yield value, which is about 15% less than the value predicted by the other two theories.

The maximum shear stress theory gives design values on the safe side and is widely used in design with ductile materials.

Part 6.2: Fatigue

6.2.1 Stress concentration

Whenever a machine component changes the shape of its cross-section, the simple stress distribution no longer holds good and the neighbourhood of the discontinuity is different. This irregularity in the stress distribution caused by abrupt changes of form is called **stress concentration.**

It occurs for all kinds of stresses in the presence of fillets, notches, holes, keyways, splines, surface roughness or scratches etc.

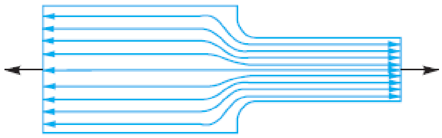

Fig. 6.2.1

In the above member with different cross-section under a tensile load, the nominal stress in the right and left hand sides will be uniform but in the region where the cross-section is changing, a re-distribution of the force within the member must take place. The maximum stress occurs at some point on the fillet and is directed parallel to the boundary at that point.

Theoretical or Form Stress Concentration Factor:

The theoretical or form stress concentration factor is defined as the ratio of the maximum stress to the nominal stress at the same section based upon net area.

$$K_t = \frac{\text{Maximum stress}}{\text{Nominal stress}}$$

The value of K_t depends upon the material and geometry of the part.

- In static loading, stress concentration in ductile materials is not so serious as in brittle materials, because in ductile materials local deformation or yielding takes place which reduces the concentration. In brittle materials, cracks may appear at these local concentrations of stress which will increase the stress over the rest of the section.
- In cyclic loading, stress concentration in ductile materials is always serious because the ductility of the material is not effective in relieving the concentration of stress caused by cracks, flaws, surface roughness, or any sharp discontinuity in the geometrical form of the member. If the stress at any point in a member is above the endurance limit of the material, a crack may develop under the action of repeated load and the crack will lead to failure of the member.

Stress Concentration due to Holes and Notches:

Consider a plate with transverse elliptical hole and subjected to a tensile load as shown in the figure.