SAMPLE OF THE STUDY MATERIAL

PART OF CHAPTER 6

Frequency Response Analysis using Bode Plot

6.1 Bode Plots:

Given open – loop transfer function of a closed – loop control system as G(S) H(S), the stability of the control system can also be determined based on its sinusoidal frequency response (obtained by substituting S = Jω). The quantities, M = 20 log_{10} |G(Jω)H(Jω)| (in db) and phase, $\phi = \angle$G(Jω)H(Jω) (in degrees) are plotted with respect to frequency on logarithmic scale ($log_{10}\omega$) in rectangular axes. The plot obtained above is called "Bode plot". G.M and PM can be found out from Bode plots, thus relative stability of closed loop control system can be assessed. Bode magnitude plot is the asymptotic approximation of actual magnitude plots by considering individual Bode magnitude plots of terms of G(Jω)H(Jω). Consider the open loop transfer function given below,

$$G(S)\,H(S) = \frac{K\left(\prod_{i=1}^{P} (1+ST_i)\right)\omega_n{}^2}{S^N\left(\prod_{J=1}^{q} (1+ST_J)\right)(S^2+2\xi\omega_n S+\omega_n{}^2)}$$

Bode magnitude plot of above can be obtained by superimposing bode magnitude plots of K, $1/_{S^N}$, $(1+ST_i)$; ∀i $\frac{1}{(1+ST_J)}$; $\forall J$ and ${\omega_n}^2/(S^2+2\xi\omega_n S+\omega_n{}^2)$.

Bode – phase plot can be obtained as any other phase response or the individual phase responses of above terms can be superimposed.

6.1.1 Bode plots of K

Bode magnitude and phase plots of constant term K are shown in figure below,

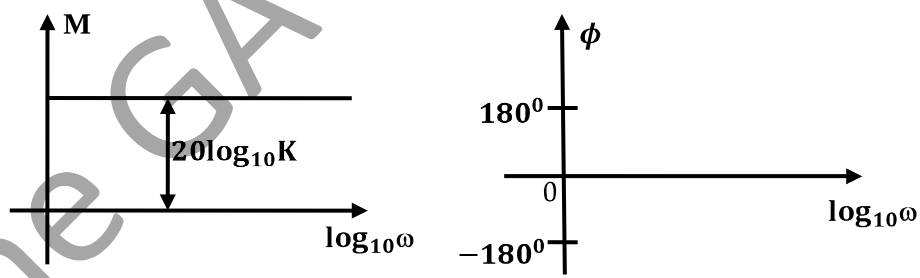


Fig 6.1 Bode plots of constant

6.1.2 Bode plots of $1/_{S^N}$:

Bode magnitude and phase plot of $1/_{S^N}$ are shown in figure below,

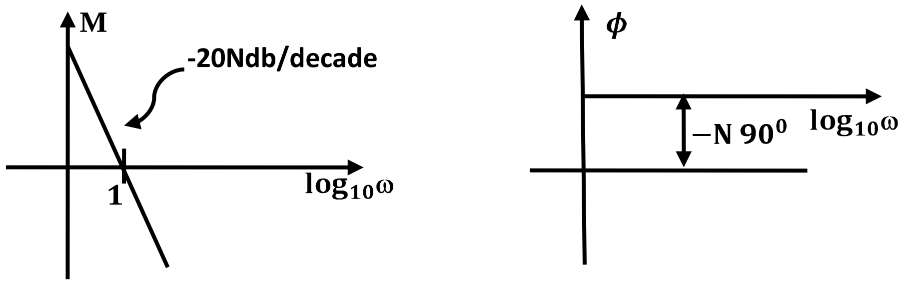


Fig 6.2 Bode plots of N^th order pole of 0.

From above we see that type of system can be found out from initial decay in Bode magnitude plot (as ω → 0).

6.1.3 Bode magnitude plot of $^{1}/_{(1 + ST)}$

Let G(s) H(s) = $\frac{1}{(1+Ts)}$

Substitute s = jω, ∴ G(Jω) H(Jω) = $\frac{1}{(1+J\omega)}$

M_{db}= 20 $\log_{10}\left(\frac{1}{\sqrt{1+(\omega T)^2}}\right)$ = - 10 log $[1 + (\omega T)^2\,]$ and $\phi = -\tan^{-1}(\omega T)$

If ω << $\frac{1}{T}$, $M \approx 0\text{db}$

If ω>> $\frac{1}{T}$, $M = -20\log(\omega T)$

Therefore, the error at the corner frequency $\omega = \frac{1}{T}$ is given as, $-10\log 2 + \log 1 = -3$ dB

The error at frequency (ω = 1 / 2T) one octave below the corner frequency is $-10\log\left(\frac{5}{4}\right) + 10\log 1 = -1$ dB

6.1.4 Bode plot of (1 + ST):-

Consider open-loop control system given below,
Let $G(S)H(S) = (1 + ST) \Rightarrow G(J\omega)H(J\omega) = (1 + J\omega T)$
M= $|G(J\omega)H(J\omega)| = 20\ \log_{10}\left(\sqrt{1+\omega^2T^2}\right)$ and $\emptyset = tan^{-1}\,(\omega T)$
If $\omega T \ll 1,\ M = 20\log_{10} 1 = 0$
If $\omega T \gg 1,\ M = 20\ \log_{10}\left(\sqrt{\omega^2T^2}\right) = \left(20\ \log_{10}\omega - 20\ \log_{10}\left(\frac{1}{T}\right)\right)$ db.
At $\omega = {}^{1}/_{T}$, error in Bode magnitude plot = 3db (as compared to original magnitude response)
Plots shown below correspond to Bode magnitude and phase plots of above system.

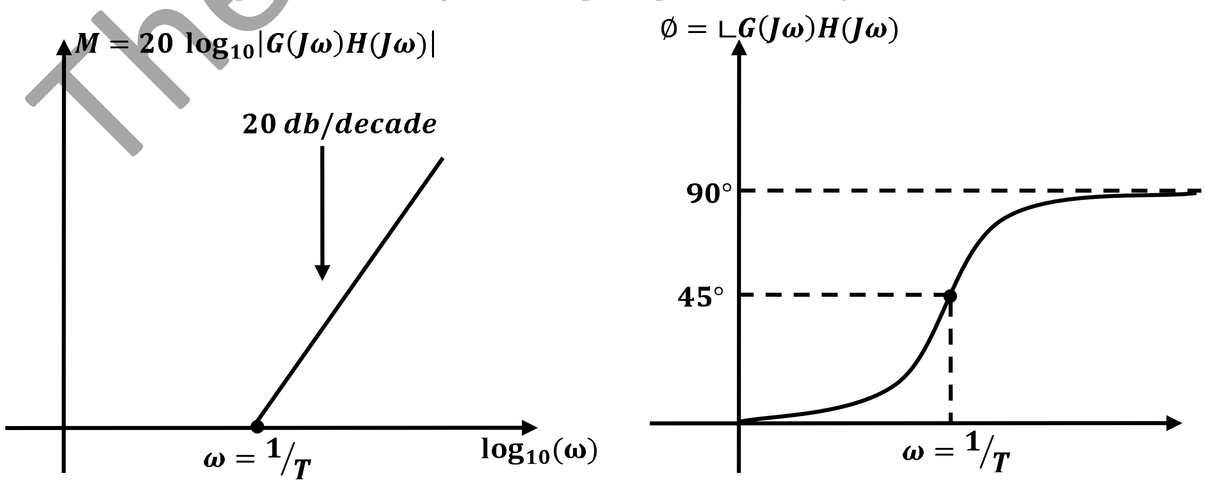


Fig. 6.3 Bode-Plots of G(S) H(S) = (1+ST)

6.1.5 Bode magnitude plot of second order control system:

Let G(s) H(s) = $\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$

Substitute s = j ω , G(Jω) H(Jω) = $\frac{\omega_n^2}{-\omega^2+j2\zeta\omega_n\omega+\omega_n^2} = \frac{1}{(1-\mu^2)^2+j2\zeta\mu}$ (where $\mu = \frac{\omega}{\omega_n}$)

∴ $M = \frac{1}{\sqrt{(1-\mu^2)^2+(2\zeta\mu)^2}}$; and $\phi = -\tan^{-1}\left[\frac{2\zeta\mu}{1-\mu^2}\right]$

$M_{dB} = -10\log\ [(1-\mu^2)^2 + 4\,\zeta^2\mu^2]$

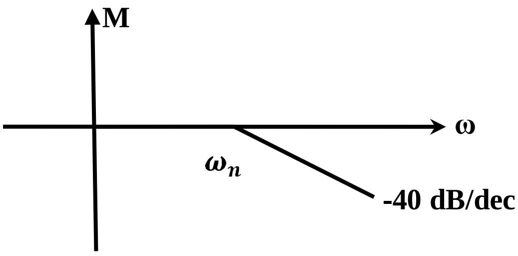


Fig. 6.4 Bode magnitude plot of II order control system

1. $\mu < 1 \Rightarrow \omega < \omega_n \Rightarrow M_{dB} \cong -10\log 1 \cong 0$ dB
2. $\mu > 1 \Rightarrow \omega > \omega_n \Rightarrow M_{dB} \cong -10\log\mu^4 \cong -40\log\mu$

The error between the actual magnitude and the asymptotic approximation is as given below.

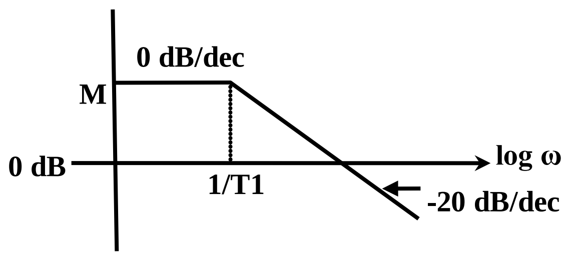
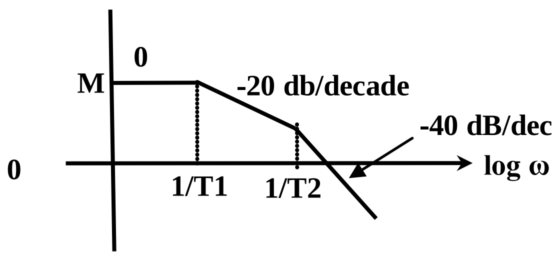
For $0 < \mu << 1$, the error is $-10\log\ [(1-\mu^2)^2 + 4\,\zeta^2\mu^2]$

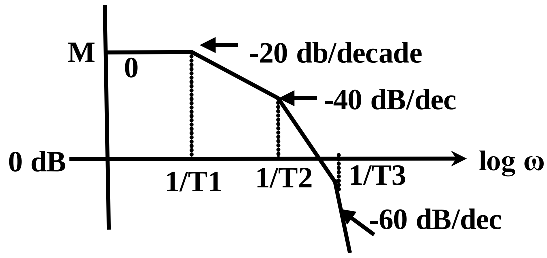
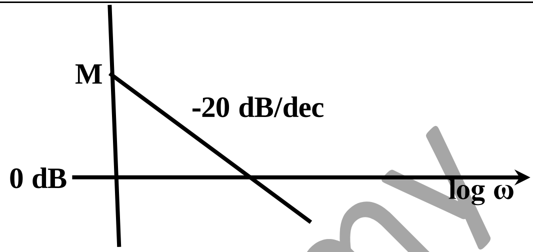
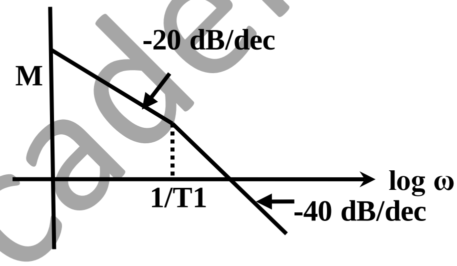
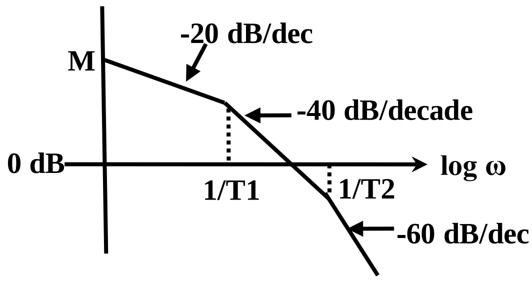
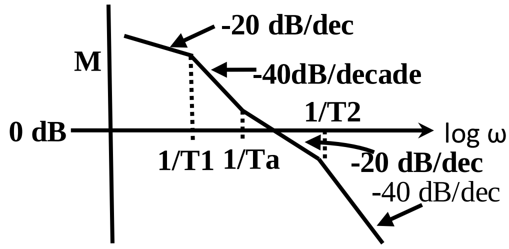
For $1 < \mu << \infty$, the error is $-10\log\ [(1-\mu^2)^2 + 4\,\zeta^2\mu^2] + 40\log\mu$

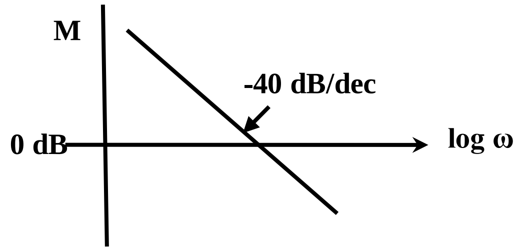
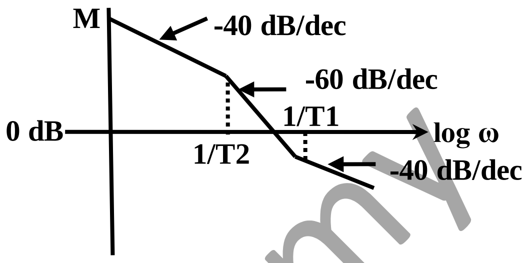
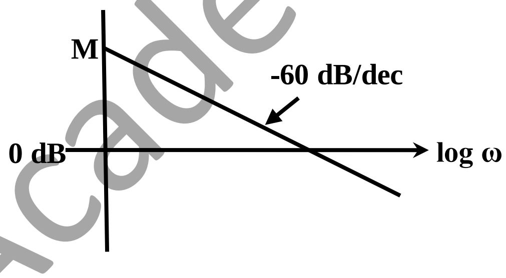
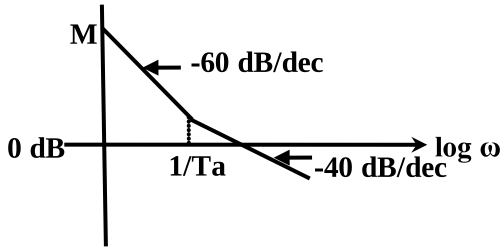
6.1.6 Bode magnitude plots for Typical Transfer Functions:

Bode magnitude and phase plots of different control systems with open loop transfer function G(s)H(s) are shown in table below,

Table 6.1.Bode magnitude plots of different control systems

S. No	G(s) H(s)	Bode Plot
1.	$\frac{K}{s\,T_1+1}$	
2.	$\frac{K}{(s\,T_1+1)(s\,T_2+1)}$ $(T_1 > T_2)$	

3.	$\frac{K}{(sT_1+1)(sT_2+1)(sT_3+1)}$ $(T_1 > T_2 > T_3)$	
4.	$\frac{K}{s}$	
5.	$\frac{K}{s(sT_1+1)}$	
6.	$\frac{K}{s(sT_1+1)(sT_2+1)}(T_1 > T_2)$	
7.	$\frac{K(sT_a+1)}{s(sT_1+1)(sT_2+1)}$ $(T_1 > T_a > T_2)$	

8.	$\frac{K}{s^2}$	
9.	$\frac{K(sT_1+1)}{s^2(sT_2+1)}\ (T_2 > T_1)$	
10.	$\frac{K}{s^3}$	
11.	$\frac{K(sT_a+1)}{s^3}$	

6.2 M & N Circles:

Constant magnitude (M) and constant phase (N) circles are discussed in detail below.

6.2.1 Constant Magnitude Loci: M-Circles

The constant magnitude contours are known as M-circles. M - circles are used to determine the magnitude response of a close-loop system using open-loop transfer function. It is applicable only for unity feedback system. The open-loop transfer function G (jω) of a unity feedback control system is a complex quantity and can be expressed as, G (jω) = x + jy

Since $M(j\omega) = \frac{C(j\omega)}{R(j\omega)} = \frac{G(j\omega)}{1+G(j\omega)} \Rightarrow M(j\omega) = \frac{x+jy}{1+x+jy} \Rightarrow M = \frac{\sqrt{x^2+y^2}}{\sqrt{(1+x)^2+y^2}}$

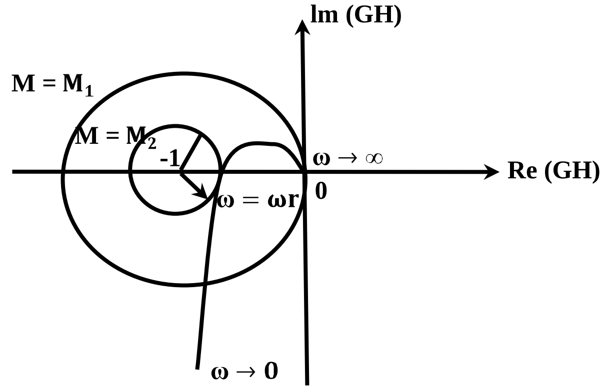


Fig. 6.5.Evaluating M-circles

On simplification, $[x - \frac{M^2}{1-M^2}]^2 + y^2 = [\frac{M}{1-M^2}]^2$

The above Eq. represents a family of circles with centre at $(\frac{M^2}{1-M^2}, 0)$, and radius as $\left|\frac{M^2}{1-M^2}\right|$. On a particular circle the value of M (magnitude of close-loop transfer function) is constant, therefore, these circles are called M-circles. The centres and radii of M-circles for different values of M are given in the following table and M-circles are drawn in the following figure.

Table 6.2.Centre and radii of M circles (for different M)

M	Centre, x = $M^2/(1-M^2)$, y = 0	Radius, r = $M/(1-M^2)$
0.5	0.33	0.67
1.0	∞	∞
1.2	-3.27	2.73
1.6	-1.64	1.03
2.0	-1.33	0.67

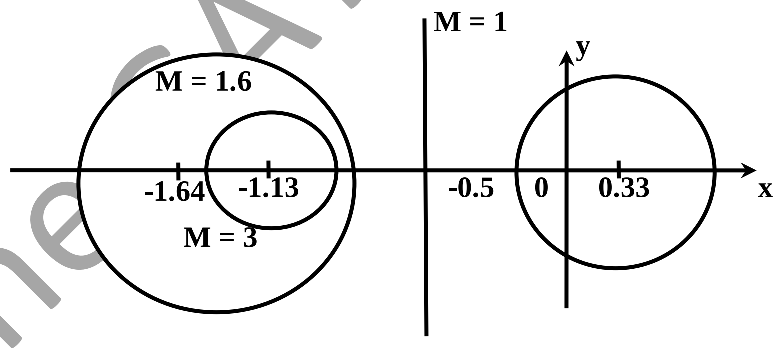


Fig. 6.6.M circles for different M

In G (jω) plane, the Nyquist plot is superimposed on M-circle and the points of intersection give the magnitude of C (jω) / R (jω) at different values of 'ω'.

6.2.2 Constant Phase Angles Loci: N-circles

The constant phase angle contours are known as N-circles. N-circles are used to determine the phase response of a close-loop system using open-loop transfer function. The phase angle of the close-loop transfer function of a unity feedback system is given by

$$\angle\frac{C(j\omega)}{R(j\omega)} = \angle\frac{x+jy}{1+x+jy}$$

The phase angle is denoted by $\phi = \tan^{-1}(y/x) - \tan^{-1}[y/(1+x)]$

$\therefore \tan\phi = \tan\{\tan^{-1}(y/x) - \tan^{-1}[y/(1+x)]\} = \frac{(y/x)-[y/(1+x)]}{1+(y/x).[y/(1+x)]} = \frac{y}{x^2+x+y^2}$

Substituting tan ϕ = N in above equation, we get $N = \tan\phi = \frac{y}{x^2+x+y^2}$

On simplification, $[x+\frac{1}{2}]^2 + [y-\frac{1}{2N}]^2 = \left[\frac{1}{4}+\frac{1}{4N^2}\right]$

For different values of N, above equation represents a family of circles with centre at x = -½ , y = 1/2N and radius as $\sqrt{\frac{1}{4}+\frac{1}{4N^2}}$. On a particular circle, the value of N or the value of phase angle of the close-loop transfer function is constant; therefore, these circles are called N-circles. The centre and radii of N-circles for different values of N are summarized in table below. Also N-circles are plotted in the figure following for different values of N.

Table 6.3.Centre and radii of N circles (for different N)

Φ	**N = tan ϕ**	**Centre x = -½ , y = 1/2N**	**Radius R = $\sqrt{1/4+1/4N^2}$**
-90^0	∞	0	0.5
-60^0	-1.732	-0.289	0.577
-50^0	-1.19	-0.42	0.656
-30^0	-0.577	0.866	1.0
-10^0	-0.176	-2.84	2.88
0^0	0	∞	∞
$+10^0$	0.176	2.84	2.88
$+30^0$	0.577	0.866	1.0
$+50^0$	1.19	0.42	0.656
$+60^0$	1.732	0.289	0.577
$+90^0$	∞	0	0.5

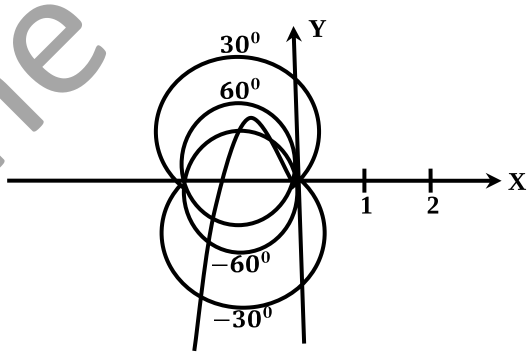


Fig.6.7.N circles for different N

6.3 Nichol's Chart

The transformation of constant – M and constant – N circles to log-magnitude and phase angle coordinates is known as the Nichols chart and the same is shown below.

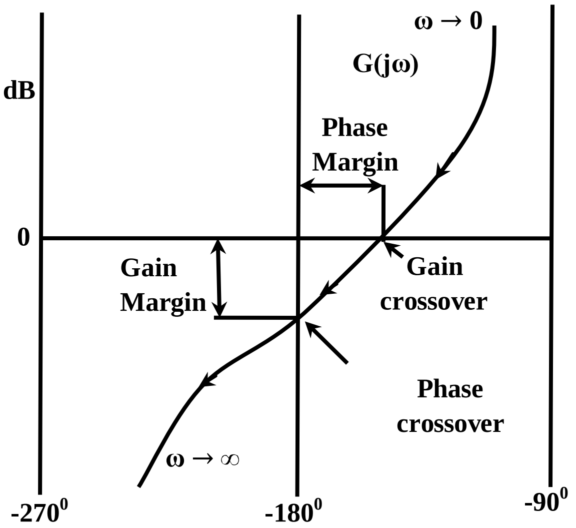


Fig.6.8. Magnitude – versus – Phase plot

Solved Examples:

Example 1: Pick the incorrect option as related to a closed loop control system if open loop transfer function G(S) H(S) = $\frac{4}{S(1+0.5S)(1+0.08S)}$

(A) Steady state decay of Bode plot is – 60 db /decade

(B) It M is Bode magnitude plot, $\frac{dM}{d\omega} \leq 0 \quad \forall\omega$

(C) $\angle G(J\omega)H(J\omega) \leq -90° \ \forall\omega$

(D) Slope of Bode magnitude plot at ω = 11 rad/sec is – 20db/decade.

Solution: As P > Z, steady state decay = -60 db/decade

Due to absence of zeroes, $\frac{dM}{d\omega} \leq 0 \ \forall\omega$

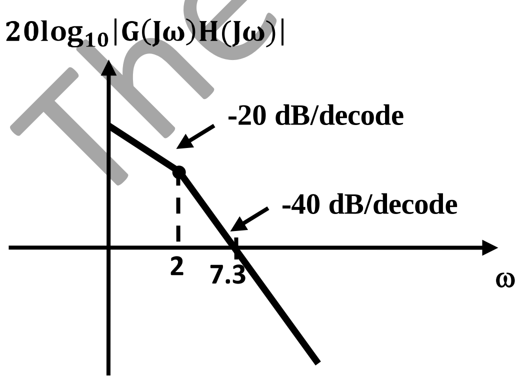
$\angle G(J\omega)H(J\omega) \leq -90°$; ∀ω

At ω = 11 rad/sec, slope of plot is -40 dB/decade. So (D) is incorrect

Hence (D)

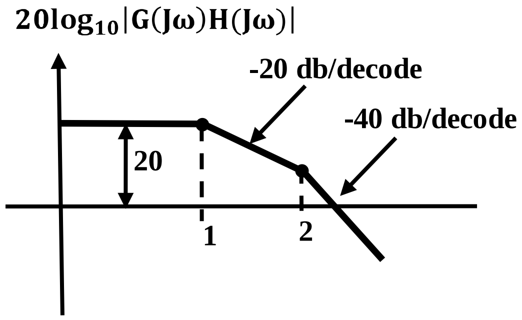
Example 2: Find the Bode magnitude plot of control system with G(S) H(S) = $\frac{4}{S(1+0.5S)}$. Also find the PM.

Solution: By superimposing Bode plots of 4, $1/_S$ and $\frac{1}{(1+0.5S)}$, Bode magnitude plot is shown below.



ω_g = 7.3rad/sec ⟹ PM = 180° + $\angle G(J\omega_g)H(J\omega_g) = 180^\circ - 90^\circ - \tan^{-1}(0.5\omega_g) = \frac{\pi}{2} - \tan^{-1}(3.65)$

Example 3: The Bode magnitude plot of a control system is given as below. Then find G(S)H(S)



Solution: G(S) H(S) = $\frac{K}{(1+ST_1)(1+ST_2)}$ and 20 $\log_{10} K = 20 \Rightarrow K = 10$

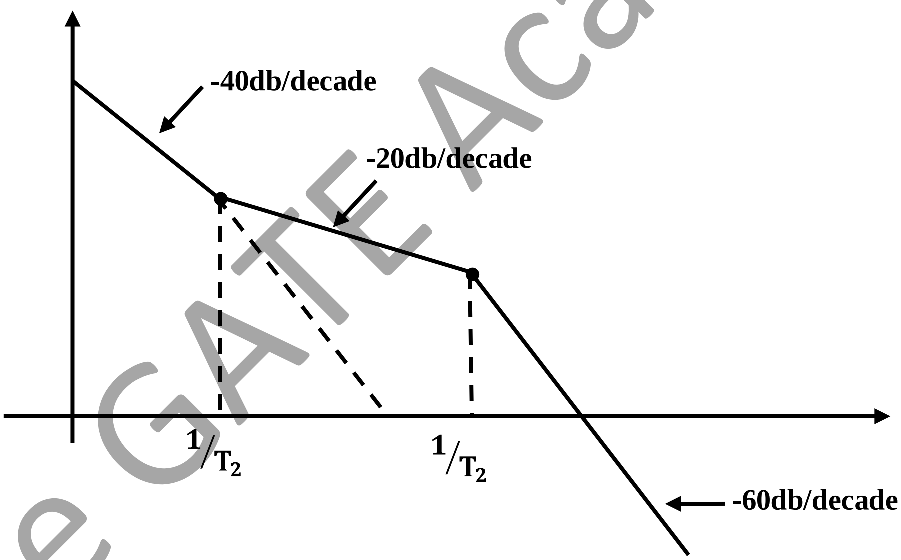
As $T_1 = 1$ and $T_2 = 0.5$, G(S) H(S) = $\frac{10}{(1+S)(1+S(0.5))}$

Example 4: The open –loop transfer function of a control system is given as G(S) H(S) = $\frac{K}{S(1+S(0.1))(1+S)}$. Find the phase cross-over frequency.

Solution: $\angle G(J\omega)H(J\omega) = \frac{-\pi}{2} - \tan^{-1}(0.1\omega) - \tan^{-1}(\omega)$

$\angle G(J\omega_p)H(J\omega_p) = -\pi \Rightarrow \omega_p = 3.4$ rad/sec.

Example 5: The open loop transfer function of a control system is given as G(S) H(S). The bode magnitude plot the system is given as below. Then pick the incorrect option as related to G(S) H(S).



(A) There is second order pole at origin

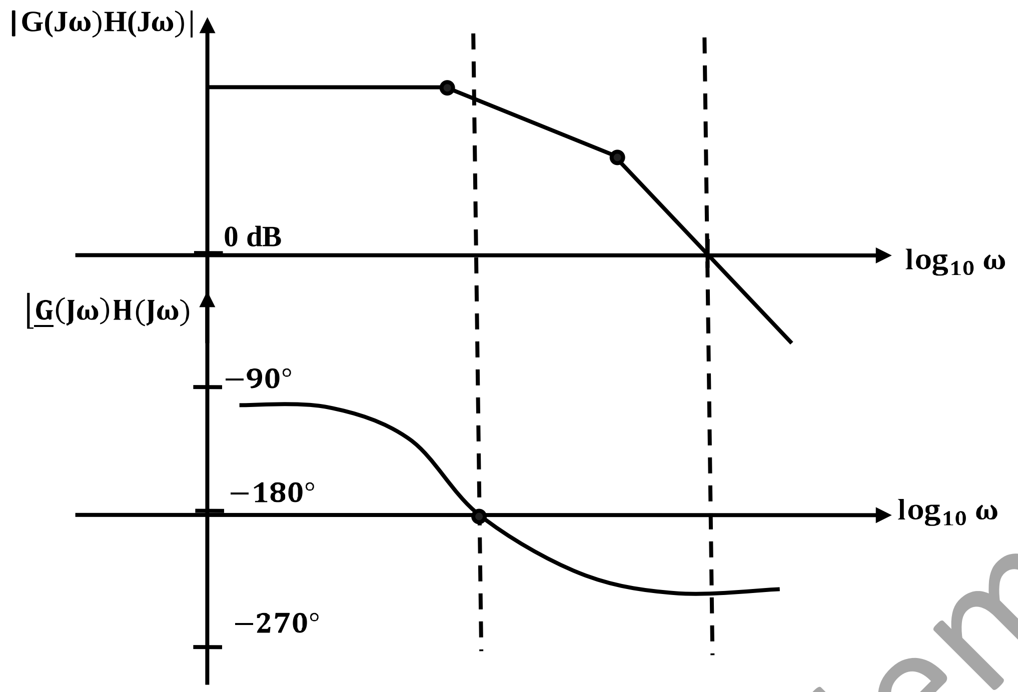
(B) There is second order zero at $^1/_{T_2}$

(C) There are two zeroes for G(S) H(S)

(D) Total number of poles

Solution: Option (C) is wrong from Bode magnitude plot properties.

Example 6: Consider the Bode magnitude and phase plats given below.



Pick the correct option as related to system given above

(A) Phase cross-over frequency is more than gain cross-over frequency
(B) System is type – 1 system
(C) G(S)H(S) has two poles
(D) G(S)H(S) has 1 zero

Solution: [Ans. C]
Due to the steady state decay of – 40 dB/decade and absence of zeros, system has two poles.

Example 7: Consider a control system of open – loop transfer function G(S)H(S) = $K/_{S\,(S+100)}$. Pick the incorrect option as related to Bode – plot of above control system

(A) Phase – plot starts with a phase of – 90°
(B) Bode – magnitude plot has a initial decay of – 20 dB / decade (as ω → 0)
(C) Steady state decay of Bode magnitude plot is – 40 dB/decade
(D) As ω → ∞ , $\angle G(J\omega)H(J\omega) \rightarrow -90^0$

Solution: [Ans. D]
$\angle G(J\omega)H(J\omega) \rightarrow -180^\circ$ as $\omega \rightarrow \infty$
(A), (B) and (C) are correct.
Hence (D) is wrong.

Example 8: Consider a control system with open−loop transfer function G(S)H(S) that has no poles at origin. The initial slope of Bode magnitude plot is _____ dB/decade

(A) −10
(B) 0
(C) 10
(D) −20

Solution: [Ans. B]

Example 9: Which of the following is not the corner frequency of Bode plot of control system with open loop transfer function, G(S)H(S) = $\frac{10}{(S+2)(S+1)(1+3S)}$

(A) $^1/_3$
(B) 10
(C) 1
(D) 2

Solution: [Ans. B]

Corner frequencies are 1, 2, $^1/_3$ rad/sec